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Metric spaces with unique pretangent spaces

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Abstract

We find necessary and sufficient conditions under which an arbitrary metric space X has a unique pretangent space at the marked point $a \in X$.

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1 Introduction

Analysis on metric spaces with no a priory smooth structure has rapidly developed the present time. This development is closely related to some generalizations of the differentiability. Important examples of such generalizations and even an axiomatics of so-called "pseudo-gradients" can be found in [1, 3–5, 9–11, 13] and respectively in [2]. In almost all above-mentioned books and papers the generalized differentiations involve an induced linear structure that makes possible to use the classical differentiations in the linear normed spaces. A new *intrinsic* approach to the introduction of the "smooth" structure for general metric spaces was proposed by O. Martio and by the first author of

the present paper in [8].

A basic technical tool in [8] is the notion of pretangent spaces at a point a of an arbitrary metric space X which were defined as factor spaces of families of sequences of points $x_n \in X$ convergent to a. In present paper we find and prove necessary and sufficient conditions under which the metric space with a marked point a has a unique pretangent space at a for every normalizing sequence $\tilde{r}$, see Definition 1.3 below.

For convenience we recall the main notions form [8], see also [6].

Let (X, d) be a metric space and let a be point of X. Fix a sequence $\tilde{r}$ of positive real numbers r_n which tend to zero. In what follows this sequence $\tilde{r}$ be called a *normalizing sequence*. Let us denote by $\tilde{X}$ the set of all sequences of points from X.

Definition 1.1. *Two sequences $\tilde{x}, \tilde{y} \in \tilde{X}$, $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$, are mutually stable (with respect to a normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$) if there is a finite limit*

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}). \tag{1.1}$$

We shall say that a family $\tilde{F} \subseteq \tilde{X}$ is *self-stable* (w.r.t. $\tilde{r}$) if every two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is *maximal self-stable* if $\tilde{F}$ is self-stable and for an arbitrary $\tilde{z} \in \tilde{X}$ either $\tilde{z} \in \tilde{F}$ or there is $\tilde{x} \in \tilde{F}$ such that $\tilde{x}$ and $\tilde{z}$ are not mutually stable.

A standard application of Zorn's Lemma leads to the following

Proposition 1.2. *Let (X, d) be a metric space and let $a \in X$. Then for every normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ there exists a maximal self-stable family $\tilde{X}_a = \tilde{X}_{a,\tilde{r}}$ such that $\tilde{a} := \{a, a, \dots\} \in \tilde{X}_a$.*

Note that the condition $\tilde{a} \in \tilde{X}_a$ implies the equality

$$\lim_{n\to\infty} d(x_n, a) = 0$$

for every $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ which belongs to $\tilde{X}_a$.

Consider a function $\tilde{d} : \tilde{X}_a \times \tilde{X}_a \to \mathbb{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (1.1). Obviously, $\tilde{d}$ is symmetric and nonnegative. Moreover, the triangle inequality for the original metric d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \le \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z}$ from $\tilde{X}_a$. Hence $(\tilde{X}_a, \tilde{d})$ is a pseudometric space.

Definition 1.3. *The pretangent space to the space X at the point a w.r.t. a normalizing sequence $\tilde{r}$ is the metric identification of the pseudometric space $(\tilde{X}_{a,\tilde{r}}, \tilde{d})$.*

Since the notion of pretangent space is basic for the present paper, we remaind this metric identification construction.

Define a relation $\sim$ on $\tilde{X}_a$ by $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. Then $\sim$ is an equivalence relation. Let us denote by $\Omega_a = \Omega_{a,\tilde{r}} = \Omega^X_{a,\tilde{r}}$ the set of equivalence classes in $\tilde{X}_a$ under the equivalence relation $\sim$. It follows from general properties of pseudometric spaces, see, for example, [12, Chapter 4, Th. 15], that if ρ is defined on Ω_a by

$$\rho(\alpha, \beta) := \tilde{d}(\tilde{x}, \tilde{y}) \tag{1.2}$$

for $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, then ρ is the well-defined metric on Ω_a. The metric identification of $(\tilde{X}_a, \tilde{d})$ is, by definition, the metric space (Ω_a, ρ).

Remark that $\Omega_{a,\tilde{r}} \neq \emptyset$ because the constant sequence $\tilde{a}$ belongs to $\tilde{X}_{a,\tilde{r}}$, see Proposition 1.2.

Let $\{n_k\}_{k\in\mathbb{N}}$ be an infinite, strictly increasing sequence of natural numbers. Let us denote by $\tilde{r}'$ the subsequence $\{r_{n_k}\}_{k\in\mathbb{N}}$ of the normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ and let $\tilde{x}' := \{x_{n_k}\}_{k\in\mathbb{N}}$ for every $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}$. It is clear that if $\tilde{x}$ and $\tilde{y}$ are mutually stable w.r.t. $\tilde{r}$, then $\tilde{x}'$ and $\tilde{y}'$ are mutually stable w.r.t. $\tilde{r}'$ and

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}'). \tag{1.3}$$

If $\tilde{X}_{a,\tilde{r}}$ is a maximal self-stable (w.r.t. $\tilde{r}$) family, then, by Zorn's Lemma, there exists a maximal self-stable (w.r.t. $\tilde{r}'$) family $\tilde{X}_{a,\tilde{r}'}$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{a,\tilde{r}}\} \subseteq \tilde{X}_{a,\tilde{r}'}.$$

Denote by $\mathrm{in}_{\tilde{r}'}$ the mapping from $\tilde{X}_{a,\tilde{r}}$ to $\tilde{X}_{a,\tilde{r}'}$ with $\mathrm{in}_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{a,\tilde{r}}$. If follows from (1.2) that after the metric identifications $\mathrm{in}_{\tilde{r}'}$ pass to an isometric embedding $\mathrm{em}'\colon \Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'}$ under which the diagram

$$\begin{tikzcd} \tilde{X}_{a,\tilde{r}} \arrow[r, "\mathrm{in}_{\tilde{r}'}"] \arrow[d, "p"'] & \tilde{X}_{a,\tilde{r}'} \arrow[d, "p'"] \\ \Omega_{a,\tilde{r}} \arrow[r, "\mathrm{em}'"] & \Omega_{a,\tilde{r}'} \end{tikzcd} \tag{1.4}$$

is commutative. Here p, p' are metric identification mappings, $p(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\}$ and $p'(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{a,\tilde{r}'} : \tilde{d}_{\tilde{r}'}(\tilde{x}, \tilde{y}) = 0\}$.

Let X and Y be two metric spaces. Recall that a map $f : X \to Y$ is called an *isometry* if f is distance-preserving and onto.

Definition 1.4. *A pretangent $\Omega_{a,\tilde{r}}$ is tangent if em$'$: $\Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'}$ is an isometry for every $\tilde{r}'$.*

Simple arguments give the following proposition.

Proposition 1.5. *Let X be a metric space with a marked point a, $\tilde{r}$ a normalizing sequence and $\tilde{X}_{a,\tilde{r}}$ a maximal self-stable family with correspondent pretangent space $\Omega_{a,\tilde{r}}$. The following statements are equivalent.*

(i) $\Omega_{a,\tilde{r}}$ is tangent.

(ii) For every subsequence $\tilde{r}'$ of the sequence $\tilde{r}$ the family $\{\tilde{x}' : \tilde{x} \in \tilde{X}_{a,\tilde{r}}\}$ is maximal self-stable w.r.t. $\tilde{r}'$.

(iii) A function $em' : \Omega_{a,\tilde{r}} \longrightarrow \Omega_{a,\tilde{r}'}$ is surjective for every $\tilde{r}'$.

(iv) A function $in'_r : \tilde{X}_{a,\tilde{r}} \longrightarrow \tilde{X}_{a,\tilde{r}'}$ is surjective for every $\tilde{r}'$.

For the proof see [6, Proposition 1.2] or [7, Proposition 1.5].

2 Conditions of uniqueness of pretangent spaces

In this section we start from the simplest example of a metric space with unique pretangent spaces.

Example 2.1. Let $X = \mathbb{R}^+ = [0, \infty[$ be the set of all non-negative, real numbers with the usual metric

$$d(x, y) = |x - y|,$$

let $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ be an arbitrary normalizing sequence and let 0 be the marked point of X. Consider a maximal self-stable family $\tilde{X}_{0,\tilde{r}}$.

Proposition 2.2. *The following statements are true.*

(i) Let $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}$. Then $\tilde{x} \in \tilde{X}_{0,\tilde{r}}$ if and only if there is $c \geq 0$ such that

$$\lim_{n\to\infty} \frac{x_n}{r_n} = c. \tag{2.1}$$

(ii) For every two $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}, \tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ from $\tilde{X}_{0,\tilde{r}}$ the equality

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0$$

holds if and only if

$$\lim_{n\to\infty} \frac{x_n}{r_n} = \lim_{n\to\infty} \frac{y_n}{r_n}.$$

(iii) The pretangent space $\Omega_{0,\tilde{r}}$ corresponding to $\tilde{X}_{0,\tilde{r}}$ is isometric to $(\mathbb{R}^+, |.,.|)$.

(iv) The pretangent space $\Omega_{0,\tilde{r}}$ is tangent.

Proof. (i) If $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}_{0,\tilde{r}}$, then there is a finite limit

$$\lim_{n\to\infty} \frac{|x_n - 0|}{r_n} = \tilde{d}(\tilde{x}, \tilde{0}).$$

Since we have $x_n = |x_n - 0|$ for all $n \in \mathbb{N}$, the limit relation (2.1) holds with $c = \tilde{d}(\tilde{x}, \tilde{0})$. Suppose that $\tilde{x}, \tilde{y} \in \tilde{X}$, $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$, $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ and there are $c_1, c_2 \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \frac{x_n}{r_n} = c_1, \ \lim_{n\to\infty} \frac{y_n}{r_n} = c_2.$$

It implies that

$$\lim_{n\to\infty} \frac{|x_n - y_n|}{r_n} = |c_1 - c_2|, \tag{2.2}$$

so $\tilde{x}$ and $\tilde{y}$ are mutually stable. It implies Statement (i).

(ii) Statement (ii) follows from Statement (i) and (2.2).

(iii) Define a function $f : \Omega_{0,\tilde{r}} \to \mathbb{R}^+$ by the rule: If $\beta \in \Omega_{0,\tilde{r}}$ and $\tilde{x} \in \beta$, then write $f(\beta) := \lim\limits_{n\to\infty} \frac{x_n}{r_n}$. Statements (i),(ii) and limit relation (2.2) imply that f is a well-defined isometry.

(iv) Let $\tilde{n} = \{n_k\}_{k\in\mathbb{N}}$ be a strictly increasing, infinite sequence of natural numbers and let $\tilde{r}' = \{r_{n_k}\}_{k\in\mathbb{N}}$ be the corresponding subsequence of the normalizing sequence $\tilde{r}$. If $\tilde{x} = \{x_k\}_{k\in\mathbb{N}} \in \tilde{X}_{0,\tilde{r}'}$ then, by Statement (i), there is $b \in \mathbb{R}^+$ such that

$$\lim_{k\to\infty} \frac{x_k}{r_{n_k}} = b.$$

Define $\tilde{y} = \{y_n\}_{n\in\mathbb{N}} \in \tilde{X}$ by the rule

$$y_n := \begin{cases} x_k & \text{if there is an element } n_k \text{ of the sequence } \tilde{n} \text{ such that } n_k = n, \\ br_n & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{y}' = \{y_{n_k}\}_{k\in\mathbb{N}} = \tilde{x}$ and

$$\lim_{n\to\infty} \frac{y_n}{r_n} = b.$$

Hence, by Statement (i), $\tilde{y}$ belongs to $\tilde{X}_{0,\tilde{r}}$. Using Proposition 1.5 we see that $\Omega_{0,\tilde{r}}$ is tangent. $\square$

Statement (i) of Proposition 2.2 shows that the space $(\mathbb{R}^+, |.,.|)$ possesses an interesting property: For every normalizing sequence $\tilde{r}$ there exists a unique pretangent space $\Omega_{0,\tilde{r}}$. The main theorem of this paper describes metric spaces which have this property.

Remark 2.3. The uniqueness in the previous paragraph and in Theorem 2.4 below is understood in the usual set-theoretical sense. Statement (i) of Proposition 2.2 implies that for $X = \mathbb{R}^+$ the family (= the set) $\tilde{X}_{a,\tilde{r}}$ is unique. Hence $\Omega_{0,\tilde{r}}$, the metric identification of $\tilde{X}_{0,\tilde{r}}$, is also unique. Since

$$\tilde{X}_{0,\tilde{r}} = \cup\{\tilde{x} \in \beta : \beta \in \Omega_{0,\tilde{r}}\},$$

i.e., the set $\tilde{X}_{0,\tilde{r}}$ is the union of all equivalence classes $\beta \in \Omega_{0,\tilde{r}}$, the uniqueness of the pretangent spaces $\Omega_{0,\tilde{r}}$ gives the uniqueness of $\tilde{X}_{0,\tilde{r}}$.

Let (X, d) be a metric space with marked point a. For each pair of nonvoid sets $C, D \subseteq X$ write

$$\Delta(C, D) := \sup\{d(x, y) : x \in C, y \in D\}, \quad \delta(C, D) := \inf\{d(x, y) : x \in C, y \in D\}$$

and write

$$A_a(r, k) := \left\{x \in X : \frac{r}{k} \le d(x, a) \le rk\right\}, \quad S_a(r) := \{x \in X : d(x, a) = r\}$$

and for every $r > 0$ and every $k \geq 1$ define

$$R_{a,X} := \left\{ r \in \mathbb{R}^+ : S_a(r) \neq \emptyset \right\}$$

and for every $\varepsilon \in]0,1[$

$$R_\varepsilon^2 := \left\{ (r,t) \in R_{a,X}^2 : r \neq 0 \neq t \text{ and } \left| \frac{r}{t} - 1 \right| \geq \varepsilon \right\}$$

where $R_{a,X}^2$ is the Cartesian product of $R_{a,X}$'s. See Fig. 1.

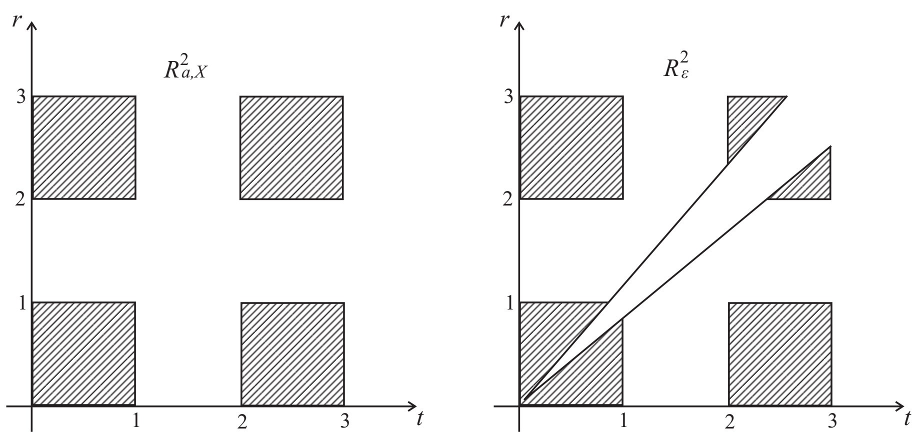

Figure 1: The sets $R_{a,X}^2$ and R_ε^2 with $R_{a,X} = [0,1] \cup [2,3]$ and $\varepsilon = \frac{1}{6}$. Non-tengential limit (2.4) is taken over the set R_ε^2.

Theorem 2.4. *Let (X,d) be a metric space and let a be a limit point of X. Then for every normalizing sequence $\tilde{r}$ there is a unique pretangent space $\Omega_{a,\tilde{r}}$ if and only if the following three conditions are satisfied simultaneously.*

(i) The limit relation

$$\lim_{k \to 1} \limsup_{r \to 0} \frac{\operatorname{diam}(A_a(r,k))}{r} = 0 \qquad r \in]0,\infty[\,, \ k \in [1,\infty[\tag{2.3}$$

holds.

(ii) We have

$$\lim_{\substack{(t,g)\to(0,0)\\(t,g)\in R_\varepsilon^2}} \frac{\Delta(S_a(g), S_a(t))}{\delta(S_a(g), S_a(t))} = 1 \tag{2.4}$$

for every $\varepsilon \in]0,1[$.

(iii) If $\{(q_n,t_n)\}_{n\in\mathbb{N}}$ is a sequence such that $(q_n,t_n) \in R_\varepsilon^2$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} (q_n,t_n) = (0,0)$$

and there is

$$\lim_{n\to\infty} \frac{q_n}{t_n} = c_0 \in [0, \infty] \,, \tag{2.5}$$

then there exists a finite limit

$$\lim_{n\to\infty} \frac{\Delta(S_a(q_n), S_a(t_n))}{|q_n - t_n|} := \varkappa_0. \tag{2.6}$$

Remark 2.5. The annulus $A_a(r,k)$ can be void in 2.3. At that time we use the convention

$$\operatorname{diam} A_r(r,k) = \operatorname{diam}(\emptyset) = 0.$$

We need the following lemma.

Lemma 2.6. *Let (X,d) be a metric space with a marked point a. A pretangent space $\Omega_{a,\tilde{r}}$ is unique for every normalizing sequence $\tilde{r}$ if and only if the implication*

$$((\tilde{x} \textit{ and } \tilde{a} \textit{ are mutually stable}) \;\&\; (\tilde{y} \textit{ and } \tilde{a} \textit{ are mutually stable}))$$

$$\Longrightarrow (\tilde{x} \textit{ and } \tilde{y} \textit{ are mutually stable}) \tag{2.7}$$

is true for every $\tilde{x}$, $\tilde{y} \in \tilde{X}$.

Proof. Suppose that (2.7) is true. Let $\tilde{X}^m_{a,\tilde{r}}$ be the set of all $\tilde{x} \in \tilde{X}$ which are mutually stable with $\tilde{a}$. It follows from (2.7) that $\tilde{X}^m_{a,\tilde{r}}$ is self-stable. Consider an arbitrary maximal self-stable $\tilde{X}_{a,\tilde{r}}$, then, by definition of $\tilde{X}_{a,\tilde{r}}$, we obtain the inclusion $\tilde{X}^m_{a,\tilde{r}} \supseteq \tilde{X}_{a,\tilde{r}}$. Since $\tilde{X}_{a,\tilde{r}}$ is maximal self-stable, we have also $\tilde{X}_{a,\tilde{r}} \supseteq \tilde{X}^m_{a,\tilde{r}}$. Hence the equality

$$\tilde{X}_{a,\tilde{r}} = \tilde{X}^m_{a,\tilde{r}}$$

holds for all $\tilde{X}_{a,\tilde{r}}$, so all $\tilde{X}_{a,\tilde{r}}$ coincide.

Now suppose that $\tilde{X}_{a,\tilde{r}}$ is unique for every $\tilde{r}$ and there are $\tilde{x}, \tilde{y} \in \tilde{X}$ and there is a normalizing sequence $\tilde{t}$ such that:

$\tilde{x}$ and $\tilde{a}$ are mutually stable;

$\tilde{y}$ and $\tilde{a}$ are mutually stable;

$\tilde{x}$ and $\tilde{y}$ are not mutually stable. By Zorn's Lemma there exist maximal self-stable families $\tilde{X}^{(1)}_{a,\tilde{t}} \supseteq \{\tilde{x}, \tilde{a}\}$ and $\tilde{X}^{(2)}_{a,\tilde{t}} \supseteq \{\tilde{y}, \tilde{a}\}$. It is clear that $\tilde{X}^{(1)}_{a,\tilde{t}} \neq \tilde{X}^{(2)}_{a,\tilde{t}}$. Hence, the uniqueness of pretangent spaces, see Remark 2.5, implies (2.7). □

Proof of Theorem 2.4. Assume that $\Omega_{a,\tilde{r}}$ is unique. We need to verify the conditions (i)–(iii).

(i) Consider a function $f : [1, \infty[\to \mathbb{R}^+$,

$$f(k) := k \limsup_{r\to 0} \frac{\operatorname{diam}(A_a(r,k))}{r}.$$

Since

$$f(k) := \limsup_{r\to 0} \frac{\operatorname{diam}(A_a(k\frac{r}{k}, k))}{\frac{r}{k}} = \limsup_{t\to 0} \frac{\operatorname{diam}(A_a(kt, k))}{t}$$

and

$$A_a(kt, k) = \{x \in X : t \le d(x, a) \le k^2 t\},$$

the function f is increasing. Since we have

$$\frac{\operatorname{diam}(A_a(r, k))}{r} \le \frac{2rk}{r} = 2k$$

for every $k \ge 1$ and all $r > 0$, the double inequality

$$0 \le f(k) \le 2k^2$$

holds. Consequently there is a finite, positive limit $\lim_{k\to 1} f(k) := c_0$. It is clear that this limit coincides with the limit in (2.3). Suppose that $c_0 > 0$. Let $\varepsilon \in \,]0, c_0[$. Then there is $k_0 > 1$ such that the double inequality

$$c_0 - \varepsilon < \limsup_{r\to 0} \frac{\operatorname{diam}(A_a(r, k))}{r_n} < c_0 + \varepsilon \tag{2.8}$$

holds for all $k \in \,]1, k_0]$. Let $\{k_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of real numbers such that all $k_n \in \,]1, k_0]$ and

$$\lim_{n\to\infty} k_n = 1. \tag{2.9}$$

Double inequality (2.8) implies that there is a sequence $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$, $r_n = r_n(k_n) > 0$, such that $\lim_{n\to 0} r_n = 0$ and

$$c_0 - \varepsilon < \frac{\operatorname{diam}(A_a(r_n, k_n))}{r_n} < c_0 + \varepsilon \tag{2.10}$$

for all $n \in \mathbb{N}$. It follows from (2.10) that there are $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ from $\tilde{X}$ such that

$$x_n, y_n \in A_a(r_n, k_n) \text{ and } \frac{d(x_n, y_n)}{r_n} \ge c_0 - \varepsilon \tag{2.11}$$

for all $n \in \mathbb{N}$. The definition of the annulus $A_a(r_n, k_n)$ and (2.11) imply that

$$\frac{d(x_n, a)}{r_n}, \frac{d(y_n, a)}{r_n} \in \left[\frac{1}{k_n}, k_n\right] \tag{2.12}$$

for all $n \in \mathbb{N}$. Define a sequence $\tilde{z} = \{z_n\}_{n\in\mathbb{N}} \in \tilde{X}$ by the rule

$$z_n := \begin{cases} x_n & \text{if} \quad n \text{ is even} \\ y_n & \text{if} \quad n \text{ is odd.} \end{cases} \tag{2.13}$$

Then it follows from (2.9), (2.12) and (2.13) that

$$\lim_{n\to\infty}\frac{d(x_n,a)}{r_n}=\lim_{n\to\infty}\frac{d(z_n,a)}{r_n}=1.$$

Moreover (2.10) and (2.12) imply that

$$\liminf_{n\to\infty}\frac{d(x_n,z_n)}{r_n}=\lim_{n\to\infty}\frac{d(x_{2n},z_{2n})}{r_{2n}}=0$$

but

$$\limsup_{n\to\infty}\frac{d(x_n,z_n)}{r_n}=\limsup_{n\to\infty}\frac{d(x_{2n+1},z_{2n+1})}{r_{2n+1}}\geq c_0-\varepsilon>0.$$

Thus $\tilde{x}$ and $\tilde{a}$ are mutually stable, $\tilde{z}$ and $\tilde{a}$ are mutually stable but $\tilde{x}$ and $\tilde{z}$ are not mutually stable (w.r.t. the normalizing sequence $\tilde{r}=\{r_n\}_{n\in\mathbb{N}}$). Hence, by Lemma 2.6, pretangent spaces to X at the point a are not unique contrary to the assumption.

(ii) Let $\varepsilon\,]0,1[$. Since

$$\Delta(C,D)\geq\delta(C,D)$$

for all nonvoid sets $C,D\subseteq X$, we have

$$1\leq\liminf_{(t,r)\to(0,0)}\frac{\Delta(S_a(r),S_a(t))}{\delta(S_a(r),S_a(r))}\leq\limsup_{(t,r)\to(0,0)}\frac{\Delta(S_a(r),S_a(t))}{\delta(S_a(r),S_a(t))}:=s_0$$

where the upper and lower limits are taken over the set R^2_ε. Hence, it is sufficient to show that $s_0=1$ in the last limit relation. Let $\tilde{r}=\{r_n\}_{n\in\mathbb{N}}$ and $\tilde{t}=\{t_n\}_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that $(r_n,t_n)\in R^2_\varepsilon$ for all $n\in\mathbb{N}$, and $\lim\limits_{n\to\infty}r_n=\lim\limits_{n\to\infty}t_n=0$ and

$$\frac{\Delta(S_a(r_n),S_a(t_n))}{\delta(S_a(r_n),S_a(t_n))}\to s_0 \tag{2.14}$$

when $n\to\infty$. Without loss of generality we may suppose that

$$0<r_n<t_n<1$$

for all $n\in\mathbb{N}$ and there is the limit

$$\lim_{n\to\infty}\frac{r_n}{t_n}:=\gamma_0. \tag{2.15}$$

First consider the case where $\gamma_0=0$. The triangle inequality implies that

$$r_n+t_n\geq\Delta(S_a(r_n),S_a(t_n))\geq\delta(S_a(r_n),S_a(t_n))\geq t_n-r_n>0. \tag{2.16}$$

Hence,

$$\frac{r_n+t_n}{t_n-r_n}\geq\frac{\Delta(S_a(r_n),S_a(t_n))}{\delta(S_a(r_n),S_a(t_n))}\geq 1, \tag{2.17}$$

from this, (2.14) and (2.16) we obtain

$$\frac{1+\gamma_0}{1-\gamma_0} \geq s_0 \geq 1. \tag{2.18}$$

Since $\gamma_0 = 0$, we see that

$$s_0 = 1.$$

Assume now that $0 < \gamma_0 < 1$. (Note that equality $\gamma_0 = 1$ contradicts the definition of the set R^2_ε.) There exist sequences $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}, \tilde{y} = \{y_n\}_{n\in\mathbb{N}}$, $\tilde{z} = \{z_n\}_{n\in\mathbb{N}}$ and $\tilde{w} = \{w_n\}_{n\in\mathbb{N}}$ from $\tilde{X}$ which satisfy the following conditions:

$$x_n \text{ and } y_n \text{ belong to } S_a(r_n) \text{ for all } n \in \mathbb{N};$$

$$z_n \text{ and } w_n \text{ belong to } S_a(t_n) \text{ for all } n \in \mathbb{N};$$

$$\lim_{n\to\infty} \frac{\Delta(S_a(r_n), S_a(t_n))}{d(x_n, z_n)} = 1; \tag{2.19}$$

$$\lim_{n\to\infty} \frac{\delta(S_a(r_n), S_a(t_n))}{d(y_n, w_n)} = 1. \tag{2.20}$$

Define new sequences $\tilde{x}^* = \{x^*_n\}_{n\in\mathbb{N}}$ and $\tilde{z}^* = \{z^*_n\}_{n\in\mathbb{N}}$ by the rules:

$$z^*_n := \begin{cases} z_n & \text{if} \quad n \text{ is even}, \\ w_n & \text{if} \quad n \text{ is odd} \end{cases} \quad \text{and} \quad x^*_n := \begin{cases} x_n & \text{if} \quad n \text{ is even}, \\ y_n & \text{if} \quad n \text{ is odd}. \end{cases}$$

Relation (2.15) and definitions of $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$, $\tilde{z}^*$, $\tilde{x}^*$ imply that

$$\lim_{n\to\infty} \frac{d(z_n, a)}{t_n} = \lim_{n\to\infty} \frac{d(w_n, a)}{t_n} = \lim_{n\to\infty} \frac{d(z^*_n, a)}{t_n} = 1$$

and

$$\lim_{n\to\infty} \frac{d(x_n, a)}{t_n} = \lim_{n\to\infty} \frac{d(y_n, a)}{t_n} = \lim_{n\to\infty} \frac{d(x^*_n, a)}{t_n} = \gamma_0 > 0.$$

Hence each from the sequences $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}, \tilde{x}^*$, $\tilde{z}^*$ is mutually stable with $\tilde{a}$. Consequently, by Lemma 2.6, there are $\tilde{d}_{\tilde{t}}(\tilde{x}, \tilde{z})$, $\tilde{d}_{\tilde{t}}(\tilde{y}, \tilde{w})$ and $\tilde{d}_{\tilde{t}}(\tilde{x}^*, \tilde{z}^*)$. Moreover (2.15), (2.16), (2.19) and (2.20) imply that

$$1 + \gamma_0 \geq \tilde{d}_{\tilde{t}}(\tilde{x}, \tilde{z}) \geq \tilde{d}_{\tilde{t}}(\tilde{y}, \tilde{w}) \geq 1 - \gamma_0 > 0.$$

It follows from (2.19), (2.20) and (2.14) that

$$0 < s_0 = \lim_{n\to\infty} \frac{\Delta(S_a(r_n), S_a(t_n))}{\delta(S_a(r_n), S_a(t_n))} = \lim_{n\to\infty} \frac{d(x_n, z_n)}{d(y_n, w_n)} = \frac{\tilde{d}_{\tilde{t}}(\tilde{x}, \tilde{z})}{\tilde{d}_{\tilde{t}}(\tilde{y}, \tilde{w})}. \tag{2.21}$$

Since $\tilde{d}_{\tilde{t}}(\tilde{y}, \tilde{w}) \neq 0$, there is a finite limit

$$\lim_{n\to\infty} \frac{d(x^*_n, z^*_n)}{d(y_n, w_n)}.$$

In particulary it follows from the definitions of $\tilde{x}^*$, $\tilde{z}^*$ that

$$\lim_{n\to\infty}\frac{d(x^*_{2n+1}, z^*_{2n+1})}{d(y_{2n+1}, w_{2n+1})} = \lim_{n\to\infty}\frac{d(y_{2n+1}, w_{2n+1})}{d(y_{2n+1}, w_{2n+1})} = 1,$$

moreover, using (2.21) we obtain

$$\lim_{n\to\infty}\frac{d(x^*_{2n}, z^*_{2n})}{d(y_{2n}, w_{2n})} = \lim_{n\to\infty}\frac{d(x_{2n}, z_{2n})}{d(y_{2n}, w_{2n})} = s_0.$$

Consequently the equality $s_0 = 1$ holds also for the case where $0 < \gamma_0 < 1$.

(iii) Let $\{(q_n, t_n)\}_{n\in\mathbb{N}}$ be a sequences of elements of R^2_ε such that $\lim\limits_{n\to\infty}(q_n, t_n) = 0$ and (2.5) holds. If in (2.5) $c_0 = 0$ or $c_0 = \infty$, then it is clear that (2.6) holds with $\varkappa_0 = 1$, so it is sufficient to take

$$0 < c_0 < \infty. \tag{2.22}$$

Consider the sequence $\tilde{q} = \{q_n\}_{n\in\mathbb{N}}$ as a normalizing sequence. Let $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ belong to $\tilde{X}$ and $d(a, x_n) = q_n$, $d(a, y_n) = t_n$ and

$$\lim_{n\to\infty}\frac{d(x_n, y_n)}{\Delta(S_a(q_n), S_a(t_n))} = 1. \tag{2.23}$$

Conditions (2.5) and (2.22) imply that there is

$$\tilde{d}_{\tilde{q}}(\tilde{y}, \tilde{a}) = \lim_{n\to\infty}\frac{d(y_n, a)}{q_n} = \frac{1}{c_0} < \infty.$$

Hence, by Lemma 2.6, there is a finite limit

$$\tilde{d}_{\tilde{q}}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty}\frac{d(x_n, y_n)}{q_n}.$$

Moreover, since $(q_n, t_n) \in R^2_\varepsilon$ for all $n \in \mathbb{N}$, we have $c_0 \neq 1$. Consequently, using (2.23) and (2.5) we obtain

$$\lim_{n\to\infty}\frac{\Delta(S_a(q_n), S_a(t_n))}{|q_n - t_n|} = \lim_{n\to\infty}\frac{d(x_n, y_n)\Delta(S_a(q_n), S_a(t_n))}{q_n\left|1 - \frac{t_n}{q_n}\right| d(x_n, y_n)} =$$

$$= \lim_{n\to\infty}\frac{d(x_n, y_n)}{q_n}\lim_{n\to\infty}\frac{1}{\left|1 - \frac{t_n}{q_n}\right|} = \frac{c_0}{|1 - c_0|}\tilde{d}_{\tilde{q}}(\tilde{x}, \tilde{y}). \tag{2.24}$$

Suppose that conditions (i)–(iii) are satisfied simultaneously. We must to prove that $\Omega_{a,\tilde{r}}$ is unique for every normalizing sequence $\tilde{r}$. Let $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ be an arbitrary normalizing sequence and let $\tilde{x} = \{x_n\}_{n\in\mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbb{N}}$ be two elements of $\tilde{X}$ such that

$$0 \le \tilde{d}(\tilde{a}, \tilde{y}) = \lim_{n\to\infty}\frac{d(a, x_n)}{r_n} < \infty$$

and

$$0 \le \tilde{d}(\tilde{a}, \tilde{y}) = \lim_{n\to\infty} \frac{d(a, y_n)}{r_n} < \infty.$$

To prove the uniqueness of $\Omega_{a,\tilde{r}}$ it is sufficient, by Lemma 2.6, to show that $\tilde{x}$ and $\tilde{y}$ are mutually stable w.r.t. $\tilde{r}$. If $\tilde{d}(\tilde{a}, \tilde{x}) = 0$, then, by the triangle inequality,

$$\limsup_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \le \lim_{n\to\infty} (\frac{d(x_n, a)}{r_n} + \frac{d(y_n, a)}{r_n}) = \tilde{d}(\tilde{a}, \tilde{y})$$

and

$$\liminf_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \ge \lim_{n\to\infty} (\frac{d(y_n, a)}{r_n} - \frac{d(x_n, a)}{r_n}) = \tilde{d}(\tilde{a}, \tilde{y}).$$

Consequently, there is a finite limit

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = d(\tilde{a}, \tilde{y}),$$

i.e., $\tilde{x}$ and $\tilde{y}$ are mutually stable. The case where $d(\tilde{a}, \tilde{y}) = 0$ is similar. Hence, without loss of generality we may assume that

$$\tilde{d}(\tilde{a}, \tilde{y}) \ne 0 \ne d(\tilde{a}, \tilde{x}).$$

Consider first the case where

$$\tilde{d}(\tilde{a}, \tilde{y}) = d(\tilde{a}, \tilde{x}) := b \ne 0.$$

This assumption implies that for every $k > 1$ there is $n_0 = n_0(k) \in \mathbb{N}$ such that the inclusion

$$A_a(br_n, k) \supseteq \{x_n, y_n\} \tag{2.25}$$

holds for all natural $n > n_0(k)$, where

$$A_a(br_n, k) = \left\{ x \in X : \frac{br_n}{k} \le d(x, a) \le kbr_n \right\}.$$

It follows from (2.25) that

$$d(x_n, y_n) \le \operatorname{diam}(A_a(br_n, k))$$

if $n > n_0(k)$. Consequently

$$\frac{1}{b} \limsup_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \le \limsup_{n\to\infty} \frac{\operatorname{diam}(A_a(br_n, k))}{br_n}.$$

Letting $k \to 1$ on the right-hand side of the last inequality and using (2.3) we see that

$$0 \le \frac{1}{b} \limsup_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \le \lim_{k\to\infty} \left(\limsup_{n\to\infty} \frac{\operatorname{diam}(A_a(br_n, k))}{br_n} \right) = 0.$$

Hence
$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = 0. \tag{2.26}$$
It implies that $\tilde{x}$ and $\tilde{y}$ are mutually stable. It still remains to show that there exists a finite limit
$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n}$$
if
$$0 \neq \tilde{d}(\tilde{x}, \tilde{a}) \neq \tilde{d}(\tilde{y}, \tilde{a}) \neq 0. \tag{2.27}$$
For convenience we write
$$q_n := d(x_n, a)\ ,\ \ t_n := d(y_n, a)$$
for all $n \in \mathbb{N}$. Condition (2.27) implies that there are $\varepsilon > 0$ and a natural number $n_0 = n_0(\varepsilon)$ such that
$$q_n \wedge t_n > 0 \text{ and } \left|\frac{q_n}{t_n} - 1\right| \geq \varepsilon \tag{2.28}$$
for all $n \geq n_0$. It is clear that
$$x_n \in S_a(q_n) \text{ and } y_n \in S_a(t_n)$$
where $S_a(q_n)$ and $S_a(t_n)$ are the spheres with the common center $a \in X$ and radiuses q_n, t_n respectively. Consequently we have the following inequalities
$$\Delta(S_a(q_n), S_a(t_n)) \geq d(x_n, y_n) \geq \delta(S_a(q_n), S_a(t_n)). \tag{2.29}$$
Limit relations (2.4) and (2.6) imply that
$$\varkappa_0 = \lim_{n\to\infty} \frac{\Delta(S_a(q_n), S_a(t_n))}{|q_n - t_n|} = \lim_{n\to\infty} \frac{\delta(S_a(q_n), S_a(t_n))}{|q_n - t_n|}.$$
Hence, using (2.29), we obtain
$$\varkappa_0 = \lim_{n\to\infty} \frac{d(x_n, y_n)}{|q_n - t_n|} = \frac{1}{\left|\tilde{d}(\tilde{x}, \tilde{a}) - \tilde{d}(\tilde{y}, \tilde{a})\right|} \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n}.$$
Hence
$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = \varkappa_0 \left|\tilde{d}(\tilde{x}, \tilde{a}) - \tilde{d}(\tilde{y}, \tilde{a})\right|, \tag{2.30}$$
i.e., $\tilde{x}$ and $\tilde{y}$ are mutually stable. □

The initial version of Theorem 2.4 was published in [7].

The following proposition will be helpful in the future.

Proposition 2.7. *Let (X, d) be a metric space with a marked point a, let $Y \subseteq X$ and let $a \in Y$. If pretangent space $\Omega^X_{a,\tilde{r}}$ is unique for every normalizing sequence $\tilde{r}$, then pretangent space $\Omega^Y_{a,\tilde{r}}$ is unique for every $\tilde{r}$.*

Proof. Apply Lemma 2.6. □

3 Future examples of metric spaces with the unique pretangent spaces

Using Example 2.1 as the simplest model we can construct some more interesting from the geometric point of view examples of metric spaces with unique tangent spaces. To this end we recall first some facts related to the structure of pretangent spaces to subspaces of metric spaces.

Let (X, d) be a metric space with a marked point a, let Y and Z be subspaces of X such that $a \in Y \cap Z$ and let $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ be a normalizing sequence.

Definition 3.1. *The subspaces Y and Z are tangent equivalent at the point a w.r.t. the normalizing sequence $\tilde{r}$ if for every $\tilde{y}_1 = \{y_n^{(1)}\}_{n\in\mathbb{N}} \in \tilde{Y}$ and every $\tilde{z}_1 = \{z_n^{(1)}\}_{n\in\mathbb{N}} \in \tilde{Z}$ with finite limits*

$$\tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{y}_1) = \lim_{n\to\infty} \frac{d(y_n^{(1)}, a)}{r_n} \quad \textit{and} \quad \tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{z}_1) = \lim_{n\to\infty} \frac{d(z_n^{(1)}, a)}{r_n}$$

there exist $\tilde{y}_2 = \{y_n^{(2)}\}_{n\in\mathbb{N}} \in \tilde{Y}$ and $\tilde{z}_2 = \{z_n^{(2)}\}_{n\in\mathbb{N}} \in \tilde{Z}$ such that

$$\lim_{n\to\infty} \frac{d(y_n^{(1)}, z_n^{(2)})}{r_n} = \lim_{n\to\infty} \frac{d(y_n^{(2)}, z_n^{(1)})}{r_n} = 0.$$

We shall say that Y and Z are *strongly tangent equivalent* at a if Y and Z are tangent equivalent at a for all normalizing sequences $\tilde{r}$.

Let $\tilde{F} \subseteq \tilde{X}$. For a normalizing sequence $\tilde{r}$ we define a family $[\tilde{F}]_Y = [\tilde{F}]_{Y,\tilde{r}}$ by the rule

$$(\tilde{y} \in [\tilde{F}]_Y) \Leftrightarrow ((\tilde{y} \in \tilde{Y}) \& (\exists\, \tilde{x} \in \tilde{F} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0)).$$

The following two lemmas were proved in [6], see also [8].

Lemma 3.2. *Let Y and Z be subspaces of a metric space X and let $\tilde{r}$ be a normalizing sequence. Suppose that Y and Z are tangent equivalent (w.r.t. $\tilde{r}$) at a point $a \in Y \cap Z$. Then following statements hold for every maximal self-stable (in $\tilde{Z}$) family $\tilde{Z}_{a,\tilde{r}}$.*

(*i*) *The family $[\tilde{Z}_{a,\tilde{r}}]_Y$ is maximal self-stable (in $\tilde{Y}$) and we have the equalities*

$$[[\tilde{Z}_{a,\tilde{r}}]_Y]_Z = \tilde{Z}_{a,\tilde{r}} = [\tilde{Z}_{a,\tilde{r}}]_Z.$$

(*ii*) *If $\Omega^Z_{a,\tilde{r}}$ and $\Omega^Y_{a,\tilde{r}}$ are metric identifications of $\tilde{Z}_{a,\tilde{r}}$ and, respectively, of $\tilde{Y}_{a,\tilde{r}} := [\tilde{Z}_{a,\tilde{r}}]_Y$, then the mapping*

$$\Omega^Z_{a,\tilde{r}} \ni \alpha \longmapsto [\alpha]_Y \in \Omega^Y_{a,\tilde{r}}$$

is an isometry. Furthermore if $\Omega^Z_{a,\tilde{r}}$ is tangent, then $\Omega^Y_{a,\tilde{r}}$ also is tangent.

(*iii*) *Moreover, if for the normalizing sequence* $\tilde{r}$ *here exists a unique maximal self-stable (in* $\tilde{Z}$*) family* $\tilde{Z}_{a,\tilde{r}} \ni \tilde{a}$*, then* $\tilde{Y}_{a,\tilde{r}} := [\tilde{Z}_{a,\tilde{r}}]_Y$ *is a unique maximal self-stable (in* $\tilde{Y}_{a,\tilde{r}}$*) family which contains* $\tilde{a}$*.*

Let Y be a subspace of a metric space (X, d). For $a \in Y$ and $t > 0$ we denote by

$$S_t^Y = S^Y(a, t) := \{y \in Y : d(a, y) = t\}$$

the sphere (in the subspace Y) with the center a and the radius t. Similarly for $a \in Z \subseteq X$ and $t > 0$ define

$$S_t^Z = S^Z(a, t) := \{z \in Z : d(a, z) = t\}.$$

Write

$$\varepsilon_a(t, Z, Y) := \sup_{z \in S_t^Z} \inf_{y \in Y} d(z, y)$$

and

$$\varepsilon_a(t) = \varepsilon_a(t, Z, Y) \vee \varepsilon_a(t, Y, Z).$$

Lemma 3.3. *Let* Y *and* Z *be subspaces of a metric space* (X, d) *and let* $a \in Y \cap Z$*. Then* Y *and* Z *are strongly tangent equivalent at the point* a *if and only if the equality*

$$\lim_{t \to 0} \frac{\varepsilon_a(t)}{t} = 0 \tag{3.1}$$

holds.

Using Proposition 2.2, Lemma 3.2 and Lemma 3.3 we can easily obtain examples of subspaces of the Euclidean space which have unique tangent spaces. The first example will be examined in details.

Example 3.4. Let $F : [0, 1] \to E^n,\ n \geq 2$, be a Jordan curve in the Euclidean space E^n, i.e., F is continuous and

$$F(t_1) \neq F(t_2)$$

for every two distinct points $t_1, t_2 \in [0, 1]$. We can write F in the coordinate form

$$F(t) = (f_1(t), \dots, f_n(t)), \qquad t \in [0, 1].$$

Suppose that all functions $f_i,\ 1 \leq i \leq n$, are differentiable at the point 0 and

$$F'(0) = (f_1'(0), \dots, f_n'(0)) \neq (0, \dots, 0).$$

(We use the one-sided derivatives here.) We claim that each pretangent space to the subspace $Y = F([0, 1]) \subseteq E^n$ at the point $a = F(0)$ is unique and tangent and isometric to $\mathbb{R}^+$ for every normalizing sequence $\tilde{r}$. Indeed, by Lemma 3.2 and by Proposition 2.2, it is sufficient to show that Y is strongly tangent equivalent to the ray

$$Z = \{(z_1(t), \dots, z_n(t)) : (z_1(t), \dots, z_n(t)) = tF'(0) + F(0),\ t \in \mathbb{R}^+\}$$

at the point $a = F(0)$.

The classical definition of the differentiability of real functions shows that limit relation (3.1) holds with these Y and Z. Hence, by Lemma 3.3, Y and Z are strongly tangent equivalent at the point $a = F(0)$.

Example 3.5. Let $f_i : [0,1] \to \mathbb{R},\ i = 1, \dots, n$, be functions such that $f_1(0) = \dots = f_n(0) = c$ where $c \in \mathbb{R}$ is a constant. Suppose all f_i have a common finite right derivative b at the point 0, $f_1'(0) = \dots = f_n'(0) = b$. Write

$$a = (0, c) \quad \text{and} \quad X = \bigcup_{i=1}^{n} \{(t, f_i(t)) : t \in [0,1]\},$$

i.e., X is an union of the graphs of the functions f_i. Let us consider X as a subspace of the Euclidean plane E^2. Then for every normalizing sequence $\tilde{r}$ a pretangent space $\tilde{\Omega}_{a,\tilde{r}}$ to the space X at the point a is unique, tangent and isometric to $\mathbb{R}^+$.

Example 3.6. Let f_1, f_2 be two functions from the precedent example. Put

$$X = \{(x, y) : f_1(x) \wedge f_2(x) \le y \le f_1(x) \vee f_2(x),\ x \in [0,1]\},$$

i.e., X is the set of points of the plane which lie between the graphs of the functions f_1 and f_2. Then for every nozmalazing sequence $\tilde{r}$ each pretangent space $\tilde{\Omega}_{a,\tilde{r}}$ to X at $a = (0, c)$ is unique, tangent and isometric to $\mathbb{R}^+$.

Example 3.7. Let α be a positive real number. Write

$$X = \{(x, y, z) \in E^3 : \sqrt{y^2 + z^2} \le x^{1+\alpha},\ x \in \mathbb{R}^+\},$$

i.e., X can be obtained by the rotation of the plane figure $\{(x, y) \in E^2 : 0 \le y \le x^{1+\alpha},\ x \in \mathbb{R}^+\}$ around the real axis. Then each pretangent space $\tilde{\Omega}_{a,\tilde{r}}$ to X at the point $a = (0, 0, 0)$ is unique, tangent and isometric to $\mathbb{R}^+$.

In the next our example we will describe the tangent space to the Cantor set C at the point 0 w.r.t. the normalizing sequence $\tilde{r} = \{\frac{1}{3^n}\}_{n \in \mathbb{N}}$. We recall the definition of the Cantor set C. Let $x \in [0,1]$ and expand x as

$$x = \sum_{n=1}^{\infty} \frac{a_{n_x}}{3^n}, \quad a_{n_x} \in \{0, 1, 2\}. \tag{3.2}$$

The Cantor set C is the set of all points from $[0,1]$ which have expansion (3.2) using only the digits 0 and 2.

Define a set C^e as the smallest subset of $\mathbb{R}$ which contains the Cantor set C and satisfies the equality

$$C^e = 3^n C^e \tag{3.3}$$

for every integer $n \in \mathbb{Z}$ where

$$3^n C^e := \{3^n x : x \in C^e\}.$$

It follows from (3.2) that a real number t belongs to C^e if and only if t has a base 3 expansion with the digits 0 and 2 only, i.e.,

$$t = \sum_{j=-\infty}^{M} a_{n_t} 3^j. \tag{3.4}$$

with $M \in \mathbb{Z}$ and $a_{n_t} \in \{0, 2\}$.

Proposition 3.8. *Let $X = C$ be the Cantor set with the usual metric $|\cdot,\cdot|$ and let $\tilde{r} = \{3^{-n}\}_{n\in\mathbb{N}}$. Then pretangent space $\Omega^X_{0,\tilde{r}}$ is unique, tangent and isometric to $(C^e, |\cdot,\cdot|)$.*

Proof. Let $\tilde{X}_{0,\tilde{r}}$ be a maximal self-stable family for which $p(\tilde{X}_{0,\tilde{r}}) = \Omega^X_{0,\tilde{r}}$, see diagram (1.4). The uniqueness of $\tilde{X}_{0,\tilde{r}}$ and of $\Omega_{0,\tilde{r}}$ follows form Proposition 2.7. As in the proof of Proposition 2.2 we see that for every $\tilde{x} = \{x_n\}_{n\in\mathbb{N}} \in \tilde{X}_{0,\tilde{r}}$ there exists a finite limit

$$\lim_{n\to\infty} \frac{x_n}{3^{-n}} = c(\tilde{x}) \tag{3.5}$$

and that the function $f : \Omega_{0,\tilde{r}} \to \mathbb{R}^+$ with

$$f(\beta) = c(\tilde{x}) \quad \text{for} \quad \tilde{x} \in \beta \in \Omega_{0,\tilde{r}} \tag{3.6}$$

is well-defined and distance-preserving. Consequently $\Omega^X_{0,\tilde{r}}$ is isometric to $(C^e, |\cdot,\cdot|)$ if the following two statements hold:

(i) $c(\tilde{x})$ belongs to C^e for every $\tilde{x} \in \tilde{X}_{0,\tilde{r}}$;

(ii) For every $t \in C^e$ there is $\tilde{x} \in \tilde{X}_{0,\tilde{r}}$ such that $c(\tilde{x}) = t$.

To prove Statement (i) note that

$$C^e = \bigcup_{i=0}^{\infty} 3^i C \tag{3.7}$$

and that for every $t > 0$ we have the equality

$$[0, t] \cap 3^i C = [0, t] \cap 3^j C \tag{3.8}$$

if $i \wedge j \geq \log_3 t$. Since C is closed, equality (3.8) and (3.7) imply that C^e also is closed. More over, using (3.2)–(3.4) we see that

$$\frac{x_n}{3^{-n}} \in C^e$$

for all $x_n \in C$ and all $n \in \mathbb{N}$. Hence $c(\tilde{x})$ belongs to C^e for every $\tilde{x} \in \tilde{X}_{0,\tilde{r}}$, that is Statement (i) follows.

Let t be an arbitrary point of C^e. Then $3^{-n}t \in C$ if $n > M$, see (3.4). Write

$$x_n := \begin{cases} 0 & \text{if } n \leq M \\ 3^{-n}t & \text{if } n > M \end{cases}$$

for $n \in \mathbb{N}$ and define $\tilde{x} := \{x_n\}_{n\in\mathbb{N}}$. It is clear that $c(\tilde{x}) = t$, so Statement (ii) is true.

It still remains to prove, that $\Omega^X_{0,\tilde{r}}$ is tangent. Let $\{n_k\}_{k\in\mathbb{N}}$ be an infinite strictly increasing sequence of natural numbers and let $\tilde{r}' := \{3^{-n_k}\}_{k\in\mathbb{N}}$. As in the proof of Statement (i) we see that the equivalence

$$\left(\tilde{x} = \{x_k\}_{k\in\mathbb{N}} \in \tilde{X}_{0\tilde{r}'}\right) \Leftrightarrow \left(\lim_{k\to\infty} \frac{x_k}{3^{-n_k}} \in C^e\right)$$

holds for every $\tilde{x} \in \tilde{X}$. By Statement (ii) we have $f(\Omega^X_{0,\tilde{r}}) = C^e$ where f is defined in (3.6). Consequently a function $em' : \Omega^X_{0,\tilde{r}} \to \Omega_{0,\tilde{r}'}$, see (1.4), is surjective. Hence, by Proposition (1.5), $\Omega_{0,\tilde{r}}$ is tangent. $\square$

Let for $x \in \mathbb{R}$

$$\varphi_0(x) := \frac{1}{3}x \quad \text{and} \quad \varphi_1(x) = \frac{1}{3}x + \frac{2}{3}. \tag{3.9}$$

It is well known that the Cantor set is the unique nonempty compact subset of $\mathbb{R}$ for which

$$X = \varphi_0(X) \cup \varphi_1(X). \tag{3.10}$$

Theorem 3.9. *Let X be the unique nonempty compact subset of $\mathbb{R}$ for which equality* (3.10) *holds, let $m = 0, 1$, and a_m be fixed points and k_m be ratios of similarities φ_m, see* (3.9)*, and $\tilde{r}_m := \{(k_m)^n\}_{n\in\mathbb{N}}$. Let us define the sets C_m and C^e_m, $m = 0, 1$ by the rules*

$$C_m = \{t - a_m : t \in C\}, \quad C^e_m = \bigcup_{j\in\mathbb{Z}} (k_m)^j C_m$$

where C is the Cantor set. Then for $m = 0, 1$ the pretangent spaces $\Omega^X_{a_m,\tilde{r}_m}$ is unique, tangent and isometric to $(C^e_m, |\cdot,\cdot|)$.

Proof. The theorem follows from Proposition 3.8 because C_0, C_1, C are isometric and C^e_0, C^e_1, C^e are isometric and $\{(k_0)^n\}_{n\in\mathbb{N}} = \{(k_1)^n\}_{n\in\mathbb{N}} = \{3^{-n}\}_{n\in\mathbb{N}}$ and $a_0 = 0 = a_1 - 1$. $\square$

Remark 3.10. Certainly, Theorem 3.9 is, on the whole, a reformulation of Proposition 3.8 but in this form the result admits generalizations for invariant sets

$$K = f_0(K) \cup f_1(K) \cup \cdots \cup f_n(K)$$

of some other iterated function systems $(f_0, \ldots, f_n)$.

In all examples above pretangent spaces $\Omega^X_{a,\tilde{r}}$ were also tangent. The following example shows that there is a metric space X for which $\Omega^X_{a,\tilde{r}}$ is unique but not tangent.

Example 3.11. Let $\tilde{r} = \{r_n\}_{n\in\mathbb{N}}$ be a sequence of strictly decreasing positive real numbers r_n with

$$\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = \infty \tag{3.11}$$

and such that $r_n > 2r_{n+1}$ for all $n \in \mathbb{N}$. Let X be a union of two countable sets $\{r_n : n \in \mathbb{N}\}$ and $\{2r_{2n} : n \in \mathbb{N}\}$ and the one-point set $\{0\}$,

$$X = \{r_n : n \in \mathbb{N}\} \cup \{2r_{2n} : n \in \mathbb{N}\} \cup \{0\}. \tag{3.12}$$

Consider the metric space $(X, |\cdot,\cdot|)$. It is clear that the sequences $\tilde{0}$ and $\tilde{x} := \{r_n\}_{n\in\mathbb{N}}$ are mutually stable w.r.t. $\tilde{r}$ and

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{0}) = 1.$$

Let $\tilde{X}_{0,\tilde{r}}$ be a unique (by Proposition 2.7) maximal self-stable family such that

$$\tilde{X}_{0,\tilde{r}} \supseteq \{0, \tilde{x}\}.$$

We claim that the pretangent space $\Omega^X_{0,\tilde{r}}$ corresponding to $\tilde{X}_{0,\tilde{r}}$ is two-point. Indeed, suppose that $\tilde{y} = \{y_n\}_{n\in\mathbb{N}} \in \tilde{X}_{0,\tilde{r}}$ and $\tilde{d}(\tilde{y}, \tilde{0}) > 0$. It is sufficient to prove that the equality

$$\tilde{d}(\tilde{x}, \tilde{y}) = 0. \tag{3.13}$$

holds. To this end, we note that (3.11) and (3.12) imply

$$\frac{y_{2n+1}}{r_{2n+1}} = 1 \quad \text{and} \quad \frac{y_{2n}}{r_{2n}} \in \{1, 2\} \tag{3.14}$$

for all sufficiently large $n \in \mathbb{N}$ because in the opposite case

$$\text{either } \lim_{n\to\infty} \frac{y_n}{r_n} = 0 \quad \text{or } \lim_{n\to\infty} \frac{y_n}{r_n} = \infty.$$

Since

$$1 = \lim_{n\to\infty} \frac{y_{2n+1}}{r_{2n+1}} = \lim_{n\to\infty} \frac{y_n}{r_n} = \lim_{n\to\infty} \frac{y_{2n}}{r_{2n}},$$

conditions (3.14) imply that

$$y_{2n} = r_{2n}$$

for sufficiently large n, Hence (3.13) follows.

Now let $\tilde{r}' := \{r_{2n}\}_{n\in\mathbb{N}}$ and $\tilde{X}_{0,\tilde{r}'}$ be a maximal self-stable family such that

$$\tilde{X}_{0,\tilde{r}'} \supseteq \{\tilde{0}, \tilde{x}, \tilde{z}\}$$

where $\tilde{x} := \{r_{2n}\}_{n\in\mathbb{N}}$ and $\tilde{z} := \{2r_{2n}\}_{n\in\mathbb{N}}$. Since

$$1 = \tilde{d}_{\tilde{r}'}(\tilde{0}, \tilde{x}) = \frac{1}{2}\tilde{d}_{\tilde{r}'}(\tilde{0}, \tilde{z}) = \tilde{d}(\tilde{x}, \tilde{y}),$$

the pretangent space $\Omega_{0,\tilde{r}'}$ corresponding to $\tilde{X}_{0,\tilde{r}'}$ contains at least three distinct points. Consequently $\Omega_{0,\tilde{r}}$ is not tangent.

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